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STABILITY ANALYSIS OF PREY PREDATOR MODEL WITH FUNCTIONAL RESPONSE

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ABSTRACT

This paper considers the prey predator model with functional response. We study the nonlinear differential equations and also existence of locally stability is discussed using the property of Jacobian. This paper deals with the question of the positive locally asymptotically stable equilibrium in a class of predator Prey systems.

KEYWORDS: Prey Predator Model, Functional Response, Local Stability, Equilibrium Point.

1. INTRODUCTION

The dynamics of two interacting populations in a patchy environment have largely been restricted to persistence and extinction analyses due to the increased complexity of global analysis. The known results on global stability are usually too general to be of any use in real mathematical or biological applications. Stabilizing or destabilizing effects of dispersions remain largely unknown due to difficulties involved in local stability analyses. Nevertheless, it is generally accepted among both Mathematicians and biologists that discrete diffusion tends to stabilize ecosystems.

The consideration in evaluating predator behavior is related to the observation that both prey and predators coexist in the same ecosystem. Therefore, a reasonable question arising is why the predators do not eliminate completely their prey.

In the 1920s, the famous Italian mathematician Vito Volterra proposed a differential equations model to describe the population dynamics of two interacting species of a predator and its prey. He hope to explain the increasing in predator fish (and corresponding decreasing in prey fish) in the Adriatic Sea during World War I. independently, this equations is studied by Volterra and Lotka [4] to describe a hypothetical chemical reaction in which the chemical concentrations oscillate and studied by Gholizade [1], in the United States. There are many species of animals in nature where one species feeds on another species. The first species and the second one are called predator and prey respectively [6, 3]. Some of the text books in this area may be found in [5, 7, 2].

In this paper, we consider the Lotka Volterra predator prey system with the following differential equations:

$$x'(t) = xg(x) - \alpha yp(x),$$

$$y'(t) = (p(x) - \beta)y. \tag{1}$$

$$x(0), y(0) > 0.$$

Here $x'(t)$ denotes the population of the prey species and $y'(t)$ denotes the population of the predator species. $p(x)$ and $g(x)$ are so-called predator and prey functional response respectively. $\alpha, \beta > 0$ are the conversion and predator's death rates, respectively. If $p(x) = \frac{mx}{a+x}$ refers to as Michaelis-Menten function or a Holling type – II function, where $m > 0$ denotes the maximal growth rate of the species, $a > 0$ is half-saturation constant. Another class of response function is Holling type-III $p(x) = \frac{mx^2}{a+x^2}$. Type III $p(x) = \frac{mx^n}{a+x^n}$ is also called the learning functional response since the curve $p(x)$ is a sigmoid type curve with an inflection point. In general the response function $p(x)$ satisfies the general hypothesis: (A) $p(x)$ is continuously differentiable function defined on $[0, \infty)$ and satisfies $p(0) = 0, p(x) > 0,$ and $\lim_{x\to\infty} p(x) = m < \infty.$ The inherent assumption in (A) is that $p(x)$ is monotonic, which is true in many predator-prey interactions. However, there is experimental and observational evidence which indicates that this need not always be the case, for example, in the cases of "inhibition" in microbial dynamics and "group defense" in population dynamics. To model such an inhibitory effect, Holling type-IV function $p(x) = \frac{mx}{a+x^2}$. found to be fit and it is simpler since it involves only two parameters. The Holling type – IV function otherwise known as Monod-Haldane function which is used in our model. The simplified Monod- Haldane or Holling type- IV functional is a modification of the Holling type-III function analyzed by [8].

The functional response is the predation rate (per predator) as a function of the prey density. In the Lotka-Volterra model (and the model above with prey density dependence) it is assumed to be a constant proportion of the prey density, i.e. a straight upward sloping line in a graph of prey killed *verses*, prey density; this is called a Type 1 functional response.

A Type 2 functional response is one in which the predation rate (fraction of prey killed) decreases as prey density increases. This is typically is caused by predators having to spend time capturing and consuming each prey, and perhaps from predators becoming satiated and ceasing to forage. On a graph of prey killed *vs*. prey density, a Type 2 functional response increases initially nearly linear fashion, but gradually slows down and eventually asymptotes at a maximum feeding rate.

In this paper, we focus on prey-predator system with type –III by introducing functional response and establish results for bounded, existence of a positively invariant and the locally asymptotical stability of coexisting interior equilibrium.

2. THE MATHEMATICAL MODEL

The model recommended in Eq.(1), where $x(t)$ and $y(t)$ represent densities of the prey species and predator species respectively; $p(x)$ and $q(x)$) are the intrinsic growth rates of the predator and prey respectively; α, β are the death rates of prey and predator respectively.

In this paper we use the following differential equations and generalize the result of [9].

If $p(x) = \frac{mx^p}{1+x^p}$ and $q(x) = ax(1-x)$. In $p(x)$ assuming $a = 1$ in general_function, that is where *a* is the half-saturation constant in the type III functional response, then Eq.(1) becomes,

$$x'(t) = x(a - bx) - \frac{\alpha m x^p y}{1 + x^p},$$
$$y'(t) = y\left(\frac{m x^p}{1 + x^p} - \beta\right). \tag{2}$$

Here a, α, β, m are all positive parameters.

Now introducing intra-specific competition, the Eq. (2) becomes

$$x'(t) = ax - bx^2 - \frac{\alpha m x^p y}{1 + x^p},$$
$$y'(t) = y\left(\frac{e\alpha m x^p}{1 + x^p} - \beta - \delta y\right). \tag{3}$$

with $x(0), y(0) > 0$ and $\alpha, \beta, \delta, m, a, b, e, p$ are all positive parameters.

Where *a* is the intrinsic growth rate of the prey population; p is the positive constant; β is the intrinsic death rate of the predator population; *b* is strength of intra-specific competition among prey species; δ is strength of intra-specific competition among predator species; *m* is direct measure of predator immunity from the prey; α is maximum attack rate of prey by predator and finally *e* represents the conversion rate.

3. EXISTENCE AND LOCAL STABILITY

Equilibria of model (3) can be obtained by equating right hand side zero.

$$ax - bx^2 - \frac{\alpha m x^p y}{1 + x^p} = 0,$$

$$y\left(\frac{e\alpha m x^p}{1 + x^p} - \beta - \delta y\right) = 0. \tag{4}$$

(i) $E_0(0,0)$, is the trivial equilibrium point that always exist.

(ii) $E_1\left(\frac{a}{b}, 0\right)$, is the axial fixed point that always exist, as the prey population grows to the carrying capacity in the absence of predation.

(iii) $E_2(x^*, y^*)$, the positive equilibrium point exists in the interior of the first quadrant if and only if there is a positive solution to the following algebraic nonlinear equations

We have the polynomial form of three and two degrees.

$$x^* = -\frac{b\delta}{e\alpha^2 m^2}x^3 + \frac{2a\delta}{e\alpha^2 m^2}x^{-p+2} + \frac{m\alpha\beta}{e\alpha^2 m^2}x + \frac{m\alpha\beta}{e\alpha^2 m^2}x^{-p+1} + \frac{a\delta}{e\alpha^2 m^2}x^{-2p+2} - \frac{b\delta}{e\alpha^2 m^2}x^{-2p+3} - \frac{2b\delta}{e\alpha^2 m^2}x^{-p+3} + \frac{a\delta}{e\alpha^2 m^2}x^2$$

If $p = 1$ it gives poly of degree 3

$$x^* = A_1 x^3 + B_1 x^2 + C_1 x + D,$$

where

$$A_1 = \frac{-b\delta}{e\alpha^2 m^2},$$

$$B_1 = \frac{a\delta}{e\alpha^2 m^2} - \frac{2b\delta}{e\alpha^2 m^2},$$

$$C_1 = \frac{2a\delta}{e\alpha^2 m^2} + \frac{m\alpha\beta}{e\alpha^2 m^2} - \frac{b\delta}{e\alpha^2 m^2},$$

$$D = \frac{a\delta}{e\alpha^2 m^2} + \frac{m\alpha\beta}{e\alpha^2 m^2}.$$

And $y^* = \frac{1}{\alpha m}\left(-bx^2 + ax + ax^{-p+1} - bx^{-p+2}\right)$

If $p = 1$ it gives polynomial of degree 2,

$$y^* = A_2 x^2 + B_2 x + C_2,$$

where

$$A_2 = -\left(\frac{b}{\alpha m}\right), \quad B_2 = \left(\frac{a}{\alpha m} - b\right), \quad C_2 = \frac{a}{\alpha m}.$$

Remark 1: There is no equilibrium point on y-axis as the predator population dies in the absence of its prey.

Lemma. For all parameters values, Eqn.(3) has fixed points, the boundary fixed point and the positive fixed point (x^*, y^*), where x^*, y^* satisfy

$$a - bx = \frac{\alpha m y x^{p-1}}{1 + x^p},$$
$$\beta + \delta y = \frac{e\alpha m y x^p}{1 + x^p}. \tag{6}$$

Now we study the stability of these fixed points. Note that the local stability of a fixed point (x, y) is determined by the modules of Eigen values of the characteristic equation at the fixed point. The Jacobian matrix J of the map (3) evaluated at any point (x, y) is given by

$$J(x,y)=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{7}$$

where

$$a_{11}=a-2bx-\frac{\alpha my}{(1+x^p)^2}(px^{p-1}),\ \ a_{12}=\frac{-\alpha mx^p}{1+x^p},$$

$$a_{21}=\frac{e\alpha ym}{(1+x^p)^2}(px^{p-1}),\ \ a_{22}=-\beta-2\delta y+\frac{e\alpha mx^p}{1+x^p}.$$

and characteristic equation of Jacobian matrix $J(x, y)$ can be written as

$$\lambda^2+p(x,y)+q(x,y)=0$$

where

$$p(x,y)=-(a_{11}+a_{12}),\ \ q(x,y)=a_{11}a_{22}-a_{12}a_{21}.$$

In order to discuss the stability of the fixed points, we also need the following lemma, which can be easily proved by the relations between roots and coefficients of a quadratic equation.

Proposition 1: The fixed point $E_1\left(\frac{a}{b},0\right)$ is locally asymptotically stable, that is sink if $a<1,\beta>\frac{em\alpha}{a^p+b^p}; E_1$ is locally unstable, that is source if $a>1,\beta<\frac{em\alpha}{a^p+b^p}; E_1$ is a saddle point if $a>1,\beta>\frac{em\alpha}{a^p+b^p}$ and E_1 is non-hyperbolic point if either $a=1$ or $\beta=\frac{em\alpha}{a^p+b^p}-1.$

Proof. One can easily see that the Jacobian matrix E_1 is

$$J_1\left(\frac{a}{b},0\right)=\begin{bmatrix} -a & \frac{-\alpha ma^p}{a^p+b^p} \\ 0 & -\beta+\frac{e\alpha ma^p}{a^p+b^p} \end{bmatrix}$$

Hence the Eigen values of the matrix are

$$|\lambda_1|=a,|\lambda_2|=\frac{e\alpha ma^p}{a^p+b^p}-\beta$$

By using Theorem, it is easy to see that, E_1 is a sink if $a<1$ and $\beta>\frac{em\alpha}{a^p+b^p}; E_1$ is a source if $a>1$ and

$\beta < \dfrac{em\alpha}{a^p + b^p}$; E_1 is a saddle if $a > 1$ and $\beta > \dfrac{em\alpha}{a^p + b^p}$; and E_1 is a saddle if either $a = 1$ or $\beta = \dfrac{em\alpha}{a^p + b^p} - 1.$

Remark 2: if $\lambda^2 - Tr(J_2) + Det(J_2) = 0,$ then the necessary and sufficient condition for linear stability is $Tr(J_2) < 0$ and $Det(J_2) > 0.$

4. STABILITY ANALYSIS

Now we investigate the local stability and bifurcations of interior fixed point E_2. The Jacobian matrix at E_2 is of the form

$$J_2(x^*, y^*) = \begin{bmatrix} a - 2bx^* - \dfrac{\alpha m y^*}{(1+x^{*p})^2}(px^{*p-1}) & -\dfrac{\alpha m x^{*p}}{1+x^{*p}} \\ e\alpha m y^* \left(\dfrac{px^{*p-1}}{(1+x^{*p})^2}\right) & -\beta - 2\delta y^* + \dfrac{e\alpha m x^*}{(1+x^{*p})^2} \end{bmatrix} \tag{8}$$

Its characteristic equation is $F(\lambda) = \lambda^2 - Tr(J_2)\lambda + Det(J_2) = 0$ where Tr is the trace and Det is the determinant of the Jacobian matrix $J(E_2)$ defined in Eq.(8), where

$$Tr(J_2) = a - 2bx^* - \frac{\alpha m y^*}{(1+x^{*p})^2}(px^{*p-1}) - \beta - 2\delta y^* + \frac{e\alpha m x^*}{(1+x^{*p})^2}$$

$$= G_1 + G_2$$

and

Det

$$(J_2) = \left(a - 2bx^* - \frac{\alpha m y^*}{(1+x^{*p})^2}(px^{*p-1})\right)\left(-\beta - 2\delta y^* + \frac{e\alpha m x^*}{(1+x^{*p})^2}\right) + \frac{e\alpha^2 m^2 x^* y^*}{(1+x^{*p})^2}(px^{*p-1})(x^{*p})$$

$$= G_1 G_2 + G_3$$

$$G_1 = a - 2bx^* - \frac{\alpha m y^*}{(1+x^{*p})^2}(px^{*p-1}),\ \ G_2 = \frac{e\alpha m x^*}{(1+x^{*p})^2} - \beta - 2\delta y^*,$$

and $G_3 = \dfrac{e\alpha^2 m^2 x^* y^*}{(1+x^{*p})^2}(px^{*p-1})(x^{*p})$

By remark 2, E_2 is stable if $G_1 + G_2 < 0$ and $G_1.G_2 + G_3 > 0.$ that is E_2 is stable if

$$\frac{e\alpha m x^*}{(1+x^{*p})^2} - \frac{\alpha m y^*}{(1+x^{*p})^2}(px^{*p-1}) - 2\delta y^* - 2bx^* < \beta - a \tag{9}$$

and

$$e < \left[\frac{(\beta + 2\delta y)\left\{(a-2bx)(1+x^p)^4 + \alpha mypx^{p-1}(1+x^p)^2\right\}}{\alpha mx\left\{(a-2bx)(1+x^p)^2\right\} - \alpha^2 m^2 yp\left\{(x^p + x^{2p})(1+x^p)^2\right\}} \right]. \tag{10}$$

If both equations (9) and (10) are satisfied, then the interior equilibrium point will be stable.

CONCLUSIONS

In this paper we investigated the complex behaviors of two species. Prey predator system as discrete time dynamical system with Holling type.

We observe that for p=1 system is stable but for p=2 system is unstable [9]. Local stability and condition of existence of equilibrium point are obtained.

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